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FUNDAMENTAL THEOREMS OF NON-LINEAR VOLRETTA-HAMMERSTEIN INTEGRAL EQUATIONS

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0 . Introduction

In this paper we shall give some fundamental theorems of a non-linear integral equation of Volterra-Hammerstein type

$$\text{(P)} \quad x(t) = f(t) + \int_0^t a(t,s)g(x(s),s)ds \quad .$$

The main purpose of this paper is to prove a local existence theorem and an extension theorem. And we shall announce some further topological properties of the family of solutions such as Kneser's property.

We first explain the notations and definitions used in this paper. For the details, see [3] and [5],

Let R^+ be the set of non-negative numbers, J be a compact interval in R^+, K be a set in R^n and $|\cdot|$ be the Euclidean norm in R^n. For each J and K we define a space $L_p(J,K)$ ($1 \leq p < \infty$) of all measurable functions $x : J \longrightarrow K$ satisfying $\|x(\cdot)\|_p < \infty$. $L_p(J,R^n)$ is known to be a Banach space, which we shall denote by $L_p(J)$. The adjoint space of $L_p(J)$ we denote by $L_p^*(J)$, and the product space $L_p^*(J) \oplus \cdots \oplus L_p^*(J)$ (with n factors) by $L_p^*(J)^n$.

We shall denote the space of linear operators on R^n by M^n.

Our standing assumptions on $f(t)$, $g(x,t)$ and $a(t,s)$ are following (I), (II) and (III).

(I) $f : R^+ \longrightarrow R^n$ is continuous

(II) Let p satisfy $1 \leqq p < \infty$. $g : R^n \times R^+ \longrightarrow R^n$ is a function such that

(1) for each $x \in R^n$, $g(x,t)$ is measurable in t

(2) for each $t \in R^+$, $g(x,t)$ is continuous in x, and

(3) for each compact set $K \subset R^n$ and each compact interval $J \subset R^+$ there exists a measurable function $m(t)$ such that

$$|g(x,t)| \leqq m(t) \quad \text{for} \quad (x,t) \in K \times J \quad \text{and} \quad \int_J m(t)^p dt < \infty .$$

(III) Let p satisfy $1 \leqq p < \infty$. $a : R^+ \times R^+ \longrightarrow M^n$ is a mapping such that

(1) for each compact interval $J \subset R^+$ and each t in R^+ the mapping $S : x(\cdot) \longrightarrow \int_J a(t,s)x(s)ds$ is a bounded linear mapping of $L_p(J)$ to R^n.

(2) the mapping $R^+ \longrightarrow L_p^*(J)^n$ defined by $t \longrightarrow a(t,\cdot)$ is continuous in the weak*-topology on $L_p^*(J)^n$.

1 . A Local Existence Theorem

R.K.Miller and G.R.Sell [5] proved an existence theorem of continuous solutions of (P) replacing the condition (III-2) by a stronger condition (III-2*) defined by

(III-2*) the mapping $R^+ \longrightarrow L_p^*(J)^n$ defined by $t \longrightarrow a(t,\cdot)$ is continuous in the strong topology on $L_p^*(J)^n$.

Their proof is completed by standard fixed point arguments in the space of continuous functions. They say in [5] that they have no counter example to show that their existence theorem is false under our weaker assumption on $a(t,s)$. If a bounded solution of (P) exists, then it is continuous even under a weaker condition (III-2). We think that our assumptions (I),(II) and (III) are natural for the existence of continuous solutions of (P).

Though we can give an existence proof of a solution by using Carathèodory approximate solutions, here we shall give a proof which uses Schauder's fixed point theorem in $L_1(J)$ with the strong topology. The essential feature of our proof is that we make use of the weak compactness of $B_p^m(J)$ in the function space $L_p(J)$ and the L_1 strong topology in the space of continuous functions.

We shall introduce the space $B_p^m(J)$ by

$$B_p^m(J) = \{\, x(\cdot) \in L_p(J) \;:\; |x(t)| \leq m(t) \text{ for a.a. } t \text{ in } J \,\}$$

and for each $x(\cdot) \in B_p^m(J)$ and $t \in J$ define the mapping T by

$$T(x)(t) = f(t) + \int_0^t a(t,s)x(s)ds .$$

Lemma 1 . *The space* $B_p^m(J)$ *is compact in the weak topology of* $L_p(J)$*, and hence the induced weak topology on* $B_p^m(J)$ *is a metric topology.*

Proof. Since $B_p^m(J)$ is bounded in $L_p(J)$, $B_p^m(J)$ is weakly sequentially compact ([1,page 294]). Furthermore, the convex set $B_p^m(J)$ is closed in the strong topology of $L_p(J)$, and hence $B_p^m(J)$ is closed in the weak topology of $L_p(J)$ ([1,page 422]). Consequently, by the Eberlein-Šmulian theorem ([1, page 430]), $B_p^m(J)$ is weakly compact. We know by ([1, page 434]) that the weak topology of $B_p^m(J)$ is a metric topology.

Lemma 2 . *The function* $t \longrightarrow T(x)(t)$ *is continuous on* J *for each fixed* $x(\cdot) \in B_p^m(J)$.

This lemma follows easily from assumptions (I) and (III). By assumption (III), we have the following Lemma 2.

Lemma 3 . *The mapping* $a(t,\cdot)$ *belongs to* $L_p^m(J)$ *for any* $t \in J$ *and* $\sup \{\|a(t,\cdot)\|_q : t \in J\} < \infty$ ($p^{-1} + q^{-1} = 1$) .

Lemma 4 . T *is a continuous mapping of* $B_p^m(J)$ *with the weak topology into* $L_1(J)$ *with the strong topology, and hence* $TB_p^m(J)$ *is compact in the strong topology on* $L_1(J)$.

Proof . By Lemma 2, $T(x)(t)$ is continuous in $t \in J$ and hence integrable on J. Therefore $Tx(\cdot)$ belongs to $L_1(J)$ for each $x(\cdot) \in B_p^m(J)$. Let $\{ x_k(\cdot) \} \subset B_p^m(J)$ be a sequence such that

$$\lim_{k\to\infty} (x_k(\cdot) - x(\cdot)) = 0 \quad \text{in the weak topology.}$$

We have by the definition of the weak topology on $B_p^m(J)$ and Lemma 3 that for each fixed t in J the mapping

$$T_t : B_p^m(J) \longrightarrow R^n \qquad \text{defined by}$$

$$T_t x(\cdot) = f(t) + \int_0^t a(t,s)x(s)ds \qquad \text{is continuous in}$$

$x(\cdot) \in B_p^m(J)$ with the weak topology. Hence we can say that $\lim_{k\to\infty} (T_t x_k(\cdot) - T_t x(\cdot)) = 0$ in R^n. Furthermore, the estimate

$$|T_t x_k(\cdot)| \leqq \max \{|f(t)| : t \in J\}$$

$$+ \sup \{\|a(t,\cdot)\|_q : t \in J\}\cdot\left(\int_J m(t)^p dt\right)^{1/p} \qquad \text{is valid for}$$

every t in J. Then $\{ T_t x_k(\cdot) \}$ is a bounded sequence of integrable functions. Hence, we have by the Lebesgue theorem that

$$\lim_{k\to\infty} \|Tx_k(\cdot) - Tx(\cdot)\|_1 = \lim_{k\to\infty}\int_J |T_t x_k(\cdot) - T_t x(\cdot)|dt = 0,$$

which shows the continuity of T in the strong topology of $L_1(J)$.

Let J be a compact interval in R^+, K be a compact set in R^n and m(t) be a function defined in (II-3) corresponding to the pair (J, K). Here we define the operator $G : L_1(J,K) \longrightarrow B_p^m(J)$ by

$$G(x)(t) = g(x(t),t) \quad .$$

Lemma 5 . *The operator* G *is continuous from* $L_1(J,K)$ *with the strong topology, into* $B_p^m(J)$ *with the strong (and hence, weak) topology.*

Proof . Let $\{ x_k(\cdot)\} \subset L_1(J,K)$ be a sequence which converges to $x(\cdot) \in L_1(J,K)$ with the strong topology. To show $\lim_{k\to\infty} Gx_k(\cdot) = Gx(\cdot)$ in $B_p^m(J)$ with the strong topology, assume the contrary. Then we can suppose that there exists an $\varepsilon_0 > 0$ such that

$$\int_J |g(x_k(s),s) - g(x(s),s)|^p ds > \varepsilon_0 \quad \text{for every } k.$$

Since $\{ x_k(\cdot)\}$ converges to $x(\cdot)$ in $L_1(J,K)$, we can also assume that $\lim_{k\to\infty} x_k(t) = x(t)$ and hence by (II-2) $\lim_{k\to\infty} g(x_k(t),t) = g(x(t),t)$ for almost all t in J. Furthermore, since $\{g(x_k(t),t)$ is bounded by the function m(t) on J, we have by the Lebesgue theorem that $\lim_{k\to\infty}\int_J |g(x_k(s),s) - g(x(s),s)|^p ds = 0$.

This contradiction proves Lemma 5 .

Now we can prove the following local existence theorem.

Theorem 6 . *Let* f, g *and* a *satisfy, respectively, assumptions* (I), (II) *and* (III) . *Then there exists an interval* $[0,\beta]$, $\beta>0$, *and a continuous function* $x : [0,\beta] \longrightarrow R^n$ *such that* (P) *is satisfied on* $[0,\beta]$.

Proof . By standard arguments we can find an interval $J = [0,\beta]$, $\beta>0$, and a compact set $K \subset R^n$ such that

$$K = \overline{\bigcup\{\ K(t)\ :\ t \in J\ \}} \quad (\text{ the closure in } R^n\ )$$

$$K(t) = \{\ p \in R^n : |\ p - f(t)\ | \leqq \delta\ \} \quad \text{and}$$

$$\delta = \sup\ \{\|a(t,\cdot)\|_q\ :\ t \in J\ \}\cdot\Big(\int_J m(t)^p dt\Big)^{1/p} \quad , \text{ where}$$

$m(t)$ is the function corresponding to the pair (J, K) in (II-3).

The set $D(J,K)$ defined by

$$D(J,K) = \{\ x(\cdot) \in L_1(J)\ :\ x(t) \in K(t) \quad \text{for a.a. } t \text{ in } J\ \}$$

is a closed and convex set in $L_1(J,K)$. By Lemma 2 and Lemma 5, the composite operator $H = T \circ G$ is continuous of $D(J,K)$ to $L_1(J)$, both with the strong topology. Furthermore, we have by Hölder's inequality

$$|\ H(x)(t) - f(t)\ | = \Big|\int_0^t a(t,s)g(x(s),s)ds\Big|$$

$$+ \quad \sup\ \{\|a(t,\cdot)\|_q\ :\ t \in J\ \}\cdot\Big(\int_J m(t)^p dt\Big)^{1/p} = \delta$$

for each $x(\cdot) \in D(J,K)$ and $t \in J$ and hence $HD(J,K) \subset D(J,K)$. Since $GD(J,K) \subset B_p^m(J)$, we have $HD(J,K) \subset TB_p^m(J)$ and hence by Lemma 4 $(\ HD(J,K))^a$ (the closure in $L_1(J)$ with the strong topology) is compact in $D(J,K)$. Therefore, by Schauder's fixed point theorem, we have an element $x(\cdot) \in D(J,K)$ such that $x(\cdot) = Hx(\cdot)$. This implies

$$x(t) = f(t) + \int_0^t a(t,s)g(x(s),s)ds \quad \text{for almost all } t \text{ in } J.$$

Since the right hand side of this equality is continuous in t., the function $x(t)$ must be continuous on J. Then we see that the above equality holds for every t in J. Thus we can obtain a continuous solution which satisfy the equation (P).

2 . An Extension Theorem

We shall define the right maximal interval J of existence for a solution $x(t)$ of (P) by the followings.

$x(t)$ is a solution of (P) on J and there does not exist an extension of $x(t)$ over an interval $\tilde{J}$ which remains a solution of (P) and J is a proper subset of $\tilde{J}$.

Theorem 7 . *Let* $x_0(t)$ *be a solution of* (P) *on some interval* J_1. *Then there exists a solution* $x(t)$ *of* (P) *on* $[0,\alpha) \supset J_1$ *that is an extension of* $x_0(t)$ *and such that* $[0,\alpha)$ *is the right maximal interval of existence for* $x(t)$. *Moreover* α *is either* ∞ *or a finite number such that* $\limsup_{t \uparrow \alpha} |x(t)| = \infty$.

Proof . Consider a given solution $x_0(t)$ on an interval J_1. Let M be the set of all extensions of $x_0(t)$, that is, an element $a \in M$ shall be represented by

$$a = \{x_a(t) : x_a(t) \text{ satisfies (P) on } J(a) \supset J_1 \text{ and } x_a|J_1 = x_0\} ,$$

where $J(a)$ is some interval and $x_a|J_1$ denotes the restriction of x_a on J_1. M is partially orderd if $a \leq a'$ means that $J(a) \subset J(a')$ and $x_{a'}|J(a) = x_a$. Let $A \subset M$ be a chain $A = \{a_\nu : \nu \in N\}$ Put $J(b) = \bigcup \{J(a_\nu) : \nu \in N\}$ and define a function $x_b(t)$, $t \in J(b)$ by $x_b(t) = x_{a_\nu}(t)$ if $t \in J(a_\nu)$. Since any two elements a_ν and $a_{\nu'}$ ($\in A$) are comparable, this function $x_b(t)$ is well-defined and satisfies (P) on $J(b)$. Therefore

$b = \{ x_b(t) : x_b(t)$ satisfies (P) on $J(b) \supset J_1$ and $x_b|J_1 = x_0\}$

belongs to M and is obviously an upper bound of A. Thus, by Zorn's lemma. M contains a maximal element

$c = \{ x(t) : x(t)$ satisfies (P) on $J \supset J_1$ and $x|J_1 = x_0 \}$.

By the definition of maximality, J is the right maximal interval of existence for $x(t)$. Either $J = [0,\infty)$ or J is a bounded interval. Suppose that J is a closed interval $[0,\alpha]$. Then the translated equation

$$y(t) = \{ f(t+\alpha) + \int_0^{\alpha} a(t+\alpha,s)g(x(s),s)ds \} + \int_0^t a(t+\alpha,s+\alpha)g(y(s),s+\alpha)ds$$

can be defined and hence by Theorem 6 this equation has a solution on some interval $[0,\delta]$. If we define $x(t+\alpha) = y(t)$, then $x(t)$ satisfies (P) on $[0,\alpha+\delta]$. This contradicts the maximality of J.

Hence J must have the form $[0,\alpha)$.

Now we shall prove the latter half of the theorem. Assume that α is not ∞ and that $x(t)$ satisfies $\limsup_{t \uparrow \alpha} |x(t)| < \infty$.

Then $K = \overline{\{ x(t) \in R^n : t \;\; [0,\alpha) \}}$ is compact. Let $m(t)$ be a function defined in (II-3) corresponding to the pair ($[0,\alpha]$, K). Let $\{t_k\}$ be a sequence such that $t_k \uparrow \alpha$ as $k \to \infty$ and $\lim_{k \to \infty} x(t_k) = x_0 \in K$. We define $x(\alpha)$ by $x(\alpha) = x_0$. Noticing that the relation

$$\left| \int_0^t a(t,s)g(x(s),s)ds - \int_0^{\alpha} a(\alpha,s)g(x(s),s)ds \right|$$

$$\leqq \left|\int_0^{\alpha} (a(t,s) - a(\alpha,s))g(x(s),s)ds\right|$$

$$+ \sup\{\|a(t,\cdot)\|_q : t \in [0,\alpha]\} \left(\int_0^{\alpha} m(t)^p dt\right)^{1/p}$$

holds for each $t < \alpha$, we have $x(t)$ satisfies (P) at $t = \alpha$ and $\lim_{t \to \alpha} x(t) = x(\alpha)$. Hence, $x(t)$ is continuous on $[0,\alpha]$ and also satisfies (P) on $[0,\alpha]$. Similarly as above, we can constract a continuous function $x(t)$ which satisfies (P) on an interval $[0,\alpha+\varepsilon]$, $\varepsilon > 0$, which contradicts the maximality of $[0,\alpha)$.

Remark. The condition $\limsup_{t \uparrow \alpha} |x(t)| = \infty$ can not be reduced to $\lim_{t \uparrow \alpha} x(t) = \infty$ (see [6] and [7]).

3. Some Further Topological Properties

In this section, we shall announce some properties of solutions without proofs. By Theorem 7, we can assume that every solution $x(t)$ of (P) is defined on some right maximal interval $[0,\alpha(x))$ for existence. Define $\beta > 0$ by $\beta = \inf\{\alpha(x) : x(\cdot)$ is a solution of (P)$\}$. We denote by F the set of all non-extendable continuous solutions of (P) and by $F(J)$ the restriction of F on a subinterval $J \subset [0,\beta)$. For each $t \in [0,\beta)$, $S(t)$ denote the cross-section in R^n, $S(t) = \{ x(t) \in R^n : x(\cdot) \in F \}$.

Lemma 8. *The closure* $\overline{\{S(t) : t \in J\}}$ *is compact in* R^n *for any compact subinterval* $J \subset [0,\beta)$.

To formulate a topological property of solution families, we need the following definition (see [1]).

A subset Q of a metric space X is called a compact R_δ iff Q is homeomorphic to the intersections of a decreasing sequence of compact absolute retracts.

Theorem 9 . *Let* f, g *and* a *satisfy, respectively, assumptions* (I) , (II) *and* (III), *and let* J *be any compact subinterval in* $[0,\beta]$. *Then the solution family* F(J) *is a compact* R_δ *in* $L_1(J)$.

This theorem can be proved by Lemma 8 and the similar arguments stated in Szufla's work [9].

Let C(J) denote the set of all continuous functions defined on an interval J. The product topology of C(J) is the topology of pointwise convergence. $F(J) \subset C(J)$ is clear. As a corollary of Theorem 9, we have the following result.

Corollary 10 . *The space* F(J) *is sequentially compact with the product topology of* C(J) *for any compact interval* $J \subset [0,\beta)$. *Moreover* S(t) *is compact and connected in* R^n *for each* $t \in [0,\beta)$.

The concrete proof of Corollary 10 will be found in author's paper [4].

If the condition (III-2) is replaced by (III-2^*) in Theorem 9 we see that F is also a compact R_δ in the Fréchet space $C[0,\beta)$. Under these stronger assumptions we have the following theorem known as Kneser's property (see [8]).

Theorem 11 . *The space* F *is compact and connected in the Fréchet space* $C[0,\beta)$. *And the cross-section* S(t) *is a continuous function of* t *in* $[0,\beta)$ *to compact and connected sets with the Hausdorff topology.*

References

[1]. F.E.Browder, C.P.Gupta, Topological degree and nonlinear mappings of analytic type in Banach space, J. Math. Anal. Appl., 26 (1969) 390-402.

[2]. N.Dunford, J.T.Schwartz, Linear operators I , interscience, 1958.

[3]. N.Kikuchi, S.Nakagiri, An Existence Theorem of Solutions of Non-Linear Integral Equations, Funkcial. Ekvac., 15 (1972) 131-138.

[4]. N.Kikuchi, S.Nakagiri, Kneser's Property of Solutiions of Non-Linear Integral Equations, Funkcial. Ekvac., 17 (1974) 57-66.

[5]. R.K.Miller, G.R.Sell, Existence, Uniqueness and Continuity of Solutions of Integral Equations, Ann. Pure. Appl., 80 (1968) 135-152.

[6]. R.K.Miller, Non-Linear Volterra Integral Equations, Mathematical Lecture Note Series, Benjamin (1971).

[7]. R.K.Miller, Some Fundamental Theory of Volterra Integral Equations. International Conference on D.E. Univ. of Southern California, Sept. (1974).

[8]. S.Nakagiri, H.Murakami, Continuities of Solutions of Non-Linear Volterra Integral Equations, Mathematica Japonicae., 20 (1975) 145-160.

[9]. S.Szufla, Solution Sets of Non-Linear Integral Equations, Funkcial. Ekvac., 17 (1974) 67-71.